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A Solution to the Optimal Pursuit Problem for Distributed Parameter Systems

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ABSTRACT

The class of systems considered in this paper includes, among others, the classical self-adjoint "Sturm–Liouville" problems of partial differential equations. The control term is an additive inhomogeneous term in the system equations and the control problem is that of minimizing the weighted sum of the "squares" of the system error and control. New results include the derivation of a sufficient condition for the optimal control, existence, and uniqueness theorems for the solution of the associated "two-point boundary-value problem," and an algorithm whose solutions converge to be optimal control. These results are illustrated by an example.

INTRODUCTION

Three broad trends may be distinguished in optimal control investigations for distributed parameter systems: approximation techniques [*1*], [*2*], exact solutions sought by variational arguments [*3*]–[*5*], and exact solutions sought by functional analysis techniques ([*6*], and this paper). The system models, the optimization problems being solved, and the required assumptions differ, of course, in these various works making exact comparisons difficult. However, restricting our attention to exact solutions, it is generally true that variational arguments in the distributed parameter context have produced only necessary conditions for the optimal control. We seek stronger results in this paper, specifically sufficiency conditions for the optimal control, existence conditions for the solution of the associated "two-point boundary-value problem," and algorithmic, that is computational, means of determining the optimal control.

In Reference [*6*], Balakrishnan has developed a general theory of optimal control problems in Banach spaces using the theory of one-parameter semigroups of linear operators and has applied some of the results to a class of control problems for distributed parameter systems. Specifically, he considers both the time-optimal and final-value control problem. In the latter, which is closer to the problem considered in this paper, the control action, which is constrained, takes place over a time interval $[0, T]$. The error between system state and desired state, however, is only of interest at time T. Here we consider the "tracking" or "pursuit" problem in a Hilbert-space setting utilizing, in part, the methods initiated in [6].

The following sections of the paper define the system model and the optimization problem being solved. After a development of the results of the paper, these results are illustrated by an example.

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1. System Model and Problem Statement

An example of the type of system and control problem we are dealing with is the inhomogeneously driven diffusion equation.

$$\left.\begin{aligned} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad \text{for } 0 < t \leqslant T \text{ and } 0 < x < 1, \\ | u(x,t) - u_0(x)| \to 0 \text{ as } t \to 0^+ \\ u(x,t) \to 0 \text{ as } x \to 0 \text{ and as } x \to 1 \text{ for } t > 0. \end{aligned}\right\} \tag{1.1}$$

Here the optimal control is that square-integrable function $f(x, t)$ which minimizes J,

$$J = \int_0^T dt \int_0^1 dx[(u_d(x,t) - u(x,t))^2 + \lambda^2 f^2(x,t)], \tag{1.2}$$

$u_d(x, t)$ is the desired system state, and λ^2 is a real parameter which weights the cost of control and error. In order to profitably generalize on this example, and thereby create a class of systems to which our analysis is to apply, we shall employ the methods of functional analysis, and in particular, the theory of one-parameter semigroups of bounded linear transformations. The basic references for these subjects are [7] and [8].

To begin with, x, in general, will be an N-dimensional variable, allowing physical processes in N space dimensions to be modeled. $u(x, t)$, for fixed x and t, will be allowed to be an m-dimensional scalar vector, so that m vector partial differential equations are included in the formulation. $u(t)$ shall be used to indicate the function $u(x, t)$ for t fixed and x variable. In general, we require that the system state $u(t)$ for $t \in [0, T]$ be in the real Hilbert space H of square-integrable functions with inner product

$$[u(t)\, v(t)]_H = \int_r dm(x)\, u^T(x,t)\, v(x,t); \tag{1.3}$$

r is the domain of the space variable x, and $m(x)$ a positive measure over the Borel sets of r, with $m(r)$ finite. The superscript T indicates the vector transpose operation.

u shall be used to indicate the function $u(x, t)$ for both x and t variable, and the real Hilbert space $H(T)$ of Bochner integrable functions is defined by

$$[u, v]_{H(T)} = \int_0^T [u(t), v(t)]_H \, dt \tag{1.4}$$

Let us require that the system state $u(t)$ be in H for $t \in [0, T]$, and that u and u_d be in $H(T)$. Further, let $f(t)$, for $t \in [0, T]$, take on values in a possible different Hilbert space F, and let f be in $F(T)$, defined analogously to $H(T)$. Then the general optimization problem is that of minimizing the quadratic functional $J(f)$ defined on $F(T)$,

$$J(f) = \| u_d - u \|_{H(T)}^2 + \lambda^2 \| f \|_{F(T)}^2 , \tag{1.5}$$

where $\| v \| = [v, v]^{1/2}$.

The system partial differential equation is modeled as a Hilbert-space-valued ordinary differential equation

$$\begin{cases} \dot{u}(t) = Au(t) + Bf(t), & \text{for } 0 < t < T. \\ \| u(t) - u_0 \|_H \to 0 \text{ as } t \to 0+ , \\ u_0 \in D(A), \end{cases} \tag{1.6}$$

B is a bounded linear operator and A is an unbounded linear operator with domain $D(A)$ in H. In the example of (1.1), H is the set of square-integrable functions over $(0, 1)$, A is the operator $\partial^2/\partial x^2$, and $D(A)$ consists of those functions in H which satisfy the boundary conditions in the x variable, and whose second derivative with respect to x is in H. In particular, we shall require that A be the infinitesimal generator of the strongly continuous semigroup $\{S(t); t \geqslant 0\}$. That is, we require that there exist a one-parameter family S of bounded linear transformations $S(t)$, mapping H into itself, such that

$$\begin{aligned} &\text{(i)} && S = \{S(t); t \geqslant 0\}; \quad S(0) = I, \text{ the identity operator,} \\ &\text{(ii)} && S(t_1 + t_2) g = S(t_1)[S(t_2) g]; \quad g \in H, t_1, t_2 \geqslant 0, \\ &\text{(iii)} && \operatorname*{Lim}_{t \to 0+} S(t) g = g, \\ &\text{(iv)} && Ag = \operatorname*{Lim}_{t \to 0+} \frac{1}{t} [S(t) - I] g \quad \text{for } g \in D(A). \end{aligned} \tag{1.7}$$

It can be shown [7] that $D(A)$ is dense in H, and that A is a closed operator.

The use of the system model (1.6) is motivated by several considerations. First, such a model includes many types of partial differential equations. In fact, such a model includes varieties of integro–partial differential equations, as discussed in [8]. Second, Balakrishnan [6] has shown that if $ABf(t)$ is Bochner-integrable over $(0, T)$, then the unique solution to (1.6) is given by

$$u(t) = S(t) u_0 + \int_0^t S(t - z) Bf(z) \, dz \tag{1.8}$$

The similarity between (1.7) and the weighting function form of solution for ordinary differential equations is striking. Indeed, the weighting function form is a specialized version of (1.7). It is because of this similarity that we can develop, in this paper, an optimal control structure parallel to, but of course more detailed than, that which exists for the corresponding class of optimization problems for ordinary differential equations. Finally, the use of the system model (1.6) is motivated by the fact that given a partial differential equation in the form (1.6), one can often ascertain the existence and uniqueness of a solution and the form of that solution (1.7) by means of the Hille–Yosida–Phillips theorem [7] without actually having to go through the often difficult process of constructing the solution.

In summary, this paper considers controling the system (1.6) so as to minimize the criterion function (1.5).

2. Necessary and Sufficient Conditions for Optimal Control and an Existence Condition for the Solution of the Associated Two-Point Boundary-Value Problem

The results of this and succeeding sections depend upon the following Lemma [9]:

Lemma. *Let G_1 and G be linear bounded operators mapping, respectively, the Hilbert spaces H_1 and H_2 into H_3, and let G^* be the adjoint to G. Consider the problem of minimizing*

$$J(f) = \| u_d - u(f)\|_{H_3}^2 + \lambda^2 \| f \|_{H_2}^2 ,$$

where

$$u = G_1 u_0 + Gf ;$$

then the optimal control, $\hat{f}$, exists uniquely in H_2 and is given by

$$\hat{f} = (G^* G + \lambda^2 I)^{-1} G^*(u_d - G_1 u_0) \tag{2.1}$$

For the results of this section, we will take the range of B in (1.6) to be in $D(A)$. Then, by the closed-graph theorem, AB is linear bounded and therefore $ABf(t)$ is Bochner-integrable, and response and control are related by

$$u(t) = S(t) u_0 + \int_0^t S(t - z) Bf(z) \, dz \in D(A). \tag{2.2}$$

Define the operators G_1 and G, with range in $H(T)$ by

$$\begin{aligned} G_1 z &= S(t) z, \qquad z \in H, \\ Gy &= \int_0^t S(t - z) y(z) \, dz, \qquad y \in H(T). \end{aligned} \tag{2.3}$$

Then, G_1 and G are bounded linear operators, and

$$u = G_1 u_0 + GBf. \tag{2.4}$$

Also

$$G^*W = \int_t^T dz\, S^*(z - t)\, w(z).$$

We note for future use that $(Gu)(t)$ and $(G^*u)(t)$ are continuous functions. Using the Lemma, the optimal control f is given by

$$(B^*G^*GB + \lambda^2 I)f = B^*G^*[u_{\mathrm{d}} - G_1 u_0]. \tag{2.5}$$

Therefore, the optimal control f has the form

$$\begin{aligned} f &= B^*q, \\ q(t) &= \frac{1}{\lambda^2} \int_t^T dz\, S^*(z - t)[u_{\mathrm{d}}(z) - u(z)], \end{aligned} \tag{2.6}$$

where $u(t)$ is the system trajectory under optimal control.

We shall now characterize this optimal control as the solution of a two-point boundary-value problem (for partial differential equations), in the form of a sufficiency condition for the optimal control.

THEOREM 2.1. A Sufficient Condition for Optimal Control. *If the "two-point boundary-value problem" (2.7) has a solution $(u(t), q(t))$, then $B^*q(t)$ is the (unique) optimal control, and $u(t)$ is the (unique) system trajectory under optimal control,*

$$\begin{cases} \dot{u}(t) = Au(t) + BB^*q(t); & \| u(t) - u_0 \|_H \to 0, \quad t \to 0^+, \quad u_0 \in D(A) \\ \dot{q}(t) = -A^*q(t) - \dfrac{1}{\lambda^2}[u_{\mathrm{d}}(t) - u(t)]; & \| q(t) \|_H \to 0, \quad t \to T^- \end{cases} \tag{2.7}$$

Proof. Let $u(t), q(t)$ be a solution to (2.7). Then $u(t) \in D(A)$, $q(t) \in D(A^*)$ and

$$\begin{aligned} \frac{d}{d\tau}(S(t - \tau)\, u(\tau)) &= -S(t - \tau)\, Au(\tau) + S(t - \tau)\, \dot{u}(\tau) \\ &= S(t - \tau)\, BB^*q(\tau), \qquad \tau \in [0, t], \end{aligned} \tag{2.8}$$

$$\begin{aligned} \frac{d}{d\tau_1}(S^*(\tau_1 - t)\, q(\tau_1)) &= S^*(\tau_1 - t)\, A^*q(\tau) + S^*(\tau_1 - t)\, \dot{q}(\tau) \\ &= -\frac{S^*(\tau_1 - t)}{\lambda^2}(u_{\mathrm{d}}(\tau_1) - u(\tau_1)), \qquad \tau_1 \in [t, T]. \end{aligned} \tag{2.9}$$

Therefore,

$$\begin{aligned} \int_0^t d\tau \left[\frac{d}{d\tau} S(t - \tau)\, u(\tau)\right] &= S(t - \tau)\, u(\tau)]_0^t \\ &= u(t) - S(t)\, u_0 = \int_0^t d\tau\, S(t - \tau)\, BB^*q(\tau) \end{aligned} \tag{2.10}$$

or

$$u(t) = S(t)\,u_0 + \int_0^t d\tau\, S(t-\tau)\,BB^*q(\tau), \tag{2.11}$$

and

$$\int_t^T d\tau_1 \left[\frac{d}{d\tau_1} S^*(\tau_1 - t)\,q(\tau_1)\right] = S^*(\tau_1 - t)\,q(\tau_1)]_t^T = -q(t)$$
$$= -\frac{1}{\lambda^2}\int_t^T S^*(\tau_1 - t)[u_d(\tau_1) - u(\tau_1)]\,d\tau_1 \tag{2.12}$$

or

$$q(t) = \frac{1}{\lambda^2}\int_t^T S^*(\tau_1 - t)[u_d(\tau_1 - u(\tau_1)]\,d\tau_1\,. \tag{2.13}$$

Then

$$u = G_3u_0 + GBB^*q, \qquad q = \lambda^{-2}G^*(u_d - u) \tag{2.13}$$

or

$$B^*q = (B^*G^*GB + \lambda^2 I)^{-1}B^*G^*[u_d - G_3u_0]. \tag{2.14}$$

Comparing this with (2.7) it is clear that B^*q is the optimal control and is therefore unique in $F(T)$. u is then the optimal system trajectory and is therefore unique in $H(T)$.

The next two theorems establish conditions sufficient to guarantee the existence of a solution to the two-point boundary-value problem (2.7).

THEOREM 2.2: An Existence Theorem for the Solution of the Two-Point Boundary-Value Problem. *If A is self-adjoint, $u_0 \in D(A)$, $u_d \in H(T)$, $u_d(t)$ continuous and $\in D(A)$ and $Au_d(t)$ Bochner-integrable, the boundary-value problem* (2.7) *has a unique solution.*

Proof. In the preceding Theorem it was shown that the following system of integral equations has the optimal control $B^*q(t) \in F(T)$ and the optimal system trajectory $u(t) \in H(T)$ for its solutions, when A is self-adjoint:

$$\begin{aligned} q(t) &= \frac{1}{\lambda^2}\int_t^T d\tau\, S(\tau - t)[u_d(\tau) - u(\tau)],\\ u(t) &= S(t)\,u_0 + \int_0^t S(t-\tau)\,BB^*q(\tau)\,d\tau. \end{aligned} \tag{2.16}$$

Here we will show that the solution $(u(t), q(t))$ to the integral equation satisfies the two-point boundary-value problems. $u(t)$ and $q(t)$ are continuous and Bochner-integrable. It was shown in [6] that the second of equations (2.16) may be differentiated to produce

$$\dot{u}(t) = Au(t) + BB^*q(t);\ \| u(t) - u_0 \| \to 0,\ t \to 0^+. \tag{2.17}$$

Thus, $u(t) \in D(A)$ and $Au(t)$, being the limit almost everywhere of the sequence of strongly measurable functions $(S(\Delta) - I/\Delta)u(t)$, is strongly measurable. From the first of Eq. (2.16),

$$\frac{q(t+\Delta) - q(t)}{\Delta} = \frac{1}{\lambda^2} \int_{t+\Delta}^{T} S(\tau - t - \Delta) \left[\frac{I - S(\Delta)}{\Delta}\right] [u_d(\tau) - u(\tau)]\, dt$$
$$- \frac{1}{\lambda^2 \Delta} \int_{t}^{t+\Delta} S(\tau - t) [u_d(\tau) - u(\tau)]\, d\tau$$
$$\| q(t) \| \to 0, \qquad t \to T^-. \tag{2.18}$$

Since $u_d(\tau) - u(\tau)$ is continuous, it is clear that the second term has $-\lambda^{-2}[u_d(t) - u(t)]$ for its limit as $\Delta \to 0$. From the "closed-graph" theorem we have that AB is bounded on F, and since $S(t - \tau)$ is bounded and strongly continuous, $S(t - \tau)ABB^*q(t)$ is Bochner-integrable on $[0, t]$. Therefore

$$Au(t) = S(t)\, Au_0 + \int_0^t S(t - \tau)\, ABB^*\, q(\tau)\, d\tau,$$
$$\int_0^T \| Au(t) \|\, dt \leqslant \int_0^T \| S(t)\, Au_0 \|\, dt \tag{2.19}$$
$$+ \int_0^T dt \int^t \| S(t - \tau)\, ABB^* q(\tau) \|\, d\tau < \infty,$$

so that the strongly measurable function $Au(t)$ is Bochner-integrable. Since $([I - S(\Delta)]/\Delta)(u_d(\tau) - u(\tau))$ converges to $-A(u_d(\tau) - u(\tau))$ and

$$\left\| \frac{I - S(\Delta)}{\Delta} (u_d(\tau) - u(\tau)) \right\| = \left\| \frac{1}{\Delta} \int_0^\Delta \frac{d}{d\sigma} [S(\sigma)(u_d(\tau) - u(\tau))]\, d\sigma \right\|$$
$$\leqslant \frac{1}{\Delta} \int_0^\Delta S(\sigma)\, A(u_d(\tau) - u(\tau))\|\, d\sigma$$
$$\leqslant K \| A(u_d(\tau) - u(\tau)) \|, \tag{2.20}$$

a Lebesgue-integrable function, it follows from the Lebesgue-dominated convergence theorem that

$$\lim_{\Delta \to 0} \int_t^T S(\tau - t) \left[\frac{I - S(\Delta)}{\Delta}\right] (u_d(\tau) - u(\tau))\, d\tau$$
$$= -\int_t^T S(\tau - t)\, A(u_d(\tau) - u(\tau))\, d\tau$$
$$= -A \int_t^T S(\tau - t)(u_d(\tau) - u(\tau))\, d\tau. \tag{2.21}$$

But

$$\begin{aligned}
&\Big\| \int_t^T S(\tau - t) \left[\frac{I - S(\Delta)}{\Delta}\right] (u_{\mathrm{d}}(\tau) - u(\tau))\, d\tau \\
&\qquad - \int_{t+\Delta}^T S(\tau - t - \Delta) \left[\frac{I - S(\Delta)}{\Delta}\right] (u_{\mathrm{d}}(\tau) - u(\tau))\, d\tau \Big\| \\
&\quad \leqslant \int_{t+\Delta}^T \| S(\tau - t - \Delta) \| \left\| \left[\frac{(I - S(\Delta))^2}{\Delta}\right] (u_{\mathrm{d}}(\tau) - u(\tau)) \right\| d\tau \\
&\quad + \int_t^{t+\Delta} \| S(\tau - t) \| \left\| \left(\frac{I - S(\Delta)}{\Delta}\right) (u_{\mathrm{d}}(\tau) - u(\tau)) \right\| d\tau \to 0 \quad \text{as} \quad \Delta \to 0. \qquad (2.22)
\end{aligned}$$

Therefore

$$\dot{q}(t) = -Aq(t) - \frac{1}{\lambda^2}[u_{\mathrm{d}}(t) - u(t)], \qquad \| q(t) \| \to 0, \qquad t \to T^-. \qquad (2.23)$$

Equations (2.17) and (2.23) show that the solution of the integral equation (2.16) satisfies the boundary-value problem.

If A is not self-adjoint, the following theorem establishes conditions sufficient to guarantee the existence of a solution to the two-point boundary-value problem (2.7). These conditions were suggested by Professor A.V. Balakrishnan.

THEOREM 2.3: An Existence Theorem for the Solution of the Two-Point Boundary-Value Problem. *If* $u_0 \in D(A)$, $u_{\mathrm{d}}(t) \in H(T)$, $u_{\mathrm{d}}(t)$ *is absolutely continuous and* $\dot{u}_{\mathrm{d}}(t)$ *is continuous in* $[0, T]$ *then the boundary-value problem* (2.7) *has a unique solution.*

Proof. Let $(u(t), q(t))$ be the known unique solution of the integral equation (2.24) which produces the optimal control. $u(t)$, $q(t)$ are in $H(T)$,

$$\begin{aligned}
q(t) &= \frac{1}{\lambda^2} \int_t^T d\tau\, S^*(\tau - t)[u_{\mathrm{d}}(\tau) - u(\tau)], \\
u(t) &= S(t)\, u_0 + \int_0^t S(t - \tau)\, BB^* q(\tau)\, d\tau.
\end{aligned} \qquad (2.24)$$

We have, from [6], that $u(t)$ satisfies the first of the boundary-value problem equations (2.7) and we shall show that $q(t)$ satisfies the second of these equations.

$q(t)$ satisfies the required boundary condition $q(T) \to 0$ as $t \to T^-$ and both $q(t)$ and $u(t)$ are continuous.

$$\begin{aligned}
\dot{u}(t) &= Au(t) + BB^* q(t) \\
&= S(t)\, Au_0 + A \int_0^t S(t - \tau)\, BB^* q(\tau)\, d\tau + BB^* q(t) \\
&= S(t)\, Au_0 + \int_0^t S(t - \tau)\, ABB^* q(\tau) + BB^* q(t), \qquad (2.25)
\end{aligned}$$

where this last step is valid because A closed and $S(t-\tau)ABB\,q(\tau)$ is Bochner-integrable on $[0, t]$, as was commented upon in the preceding theorem. Continuing,

$$\dot{u}(t) = S(t)\,Au_0 + \int_0^t S(t-\tau)\,ABB^*q(\tau)\,d\tau$$
$$+ \frac{BB^*}{\lambda^2}\int_t^T d\tau\, S^*(\tau - t)[u_d(\tau) - u(\tau)]. \tag{2.26}$$

Each of the above terms is continuous: The first because $u_0 \in D(A)$ and $S(t)$ strongly continuous, the second because it is in the range of G and the third because it is in the range of G^*. Therefore, $(u_d(t) - u(t))$, which we shall call $\omega(t)$, is continuous and has a continuous derivative.

$$\frac{q(t+\Delta) - q(t)}{\Delta} = \frac{1}{\lambda^2\Delta}\Big[\int_{t+\Delta}^T \frac{1}{\lambda^2\Delta}\int_{t+\Delta}^T d\tau\, S^*(\tau - t - \Delta)\,\omega(\tau)\,d\tau$$
$$- \int_t^T d\tau\, S^*(\tau - t)\,\omega(\tau)\Big]$$
$$= -\frac{1}{\lambda^2\Delta}[S^*(\Delta) - I]\int_{t+\Delta}^T S^*(\tau - t - \Delta)\,\omega(\tau)\,d\tau$$
$$- \frac{1}{\lambda^2\Delta}\int_t^{t+\Delta} d\tau\, S^*(\tau - t)\,\omega(\tau). \tag{2.27}$$

Since $S^*(\tau - t)\omega(\tau)$ is continuous the second term has $-\lambda^{-2}\omega(t)$ for a limit as $\Delta \to 0$. The first term has the same limit as

$$\frac{(I - S^*(\Delta))}{\lambda^2\Delta}\int_t^T S^*(\tau - t)\,\omega(\tau)\,d\tau$$

for

$$\Big(\frac{I - S^*(\Delta)}{\lambda^2\Delta}\Big)\Big[\int_{t+\Delta}^T S^*(\tau - t - \Delta)\,\omega(\tau)\,d\tau - \int_t^T S^*(\tau - t)\,\omega(\tau)\,d\tau\Big]$$
$$= \frac{I - S^*(\Delta)}{\lambda^2}\int_t^{T-\Delta} S^*(\tau - t)\Big(\frac{\omega(\tau+\Delta) - \omega(\tau)}{\Delta}\Big)\,d\tau$$
$$- \Big(\frac{I - S^*(\Delta)}{\lambda^2\Delta}\Big)\int_{T-\Delta}^T S^*(\tau - t)\,\omega(\tau)\,d\tau. \tag{2.28}$$

Now, letting $\eta(\Delta) \in H$ such that $\|\eta(\Delta)\| \to 0$ with $\Delta \to 0$,

$$\Big(\frac{I - S^*(\Delta)}{\lambda^2\Delta}\Big)\int_{T-\Delta}^T S^*(\tau - t)\,\omega(\tau)\,d\tau$$
$$= (I - S^*(\Delta))(S^*(T - t)\,\omega(T) + \eta(\Delta)) \to 0 \quad \text{as} \quad \Delta \to 0, \tag{2.29}$$

as a result of the strong continuity of $S^*(\Delta)$. Also, since

$$\left\| \frac{\omega(\tau+\Delta)-\omega(\tau)}{\Delta} \right\| = \left\| \frac{1}{\Delta} \int_0^\Delta \dot{\omega}(\tau+\sigma)\, d\sigma \right\| \leqslant \sup \| \dot{\omega}(\sigma) \|_{\sigma\in[t,T]} \leqslant K, \tag{2.30}$$

$$\begin{aligned} &\left\| \int_t^{T-\Delta} S^*(\tau - t) \left(\frac{\omega(\tau+\Delta)-\omega(\tau)}{\Delta} \right) dt \right\| \\ &\quad \leqslant \int_t^{T-\Delta} \| S^*(\tau - t) \| \left\| \frac{\omega(\tau+\Delta)-\omega(\tau)}{\Delta} \right\| d\tau \\ &\quad \leqslant K' \int_t^T \| S^*(\tau - t) \| \, d\tau < k', \end{aligned} \tag{2.31}$$

and therefore

$$\left(\frac{I - S^*(\Delta)}{\lambda} \right) \int_t^{T-\Delta} S^*(\tau - t) \left(\frac{\omega(\tau+\Delta)-\omega(\tau)}{\Delta} \right) d\tau \to 0 \tag{2.32}$$

Thus

$$\frac{q(t+\Delta)-q(t)}{\Delta} \to \frac{I - S^*(\Delta)}{\lambda^2 \Delta} \int_t^T S^*(\tau - t)\, \omega(\tau)\, d\tau - \frac{1}{\lambda^2}\, \omega(t) \tag{2.33}$$

Now, if $\int_t^T S^*(\tau - t)\, \omega(t)\, dt$ is in $D(A^*)$, the first term has $-(A^*/\lambda^2) \int_t^T S^*(\tau - t)\, \omega(\tau)\, d\tau$ for its limit. And this is so, for

$$\begin{aligned} &\left(\frac{I - S^*(\Delta)}{\Delta} \right) \int_t^T S^*(\tau - t)\, \omega(\tau)\, d\tau \\ &\quad = \int_t^{T-\Delta} S^*(\tau + \Delta - t) \left[\frac{\omega(\tau+\Delta)-\omega(\tau)}{\Delta} \right] d\tau \\ &\qquad + \frac{1}{\Delta} \int_t^{t+\Delta} S^*(\tau - t)\, \omega(\tau)\, d\tau + \frac{1}{\Delta} \int_{T-\Delta}^T S^*(\tau + \Delta - t)\, \omega(\tau)\, d\tau. \end{aligned}$$

The second and third terms have $\omega(t)$ and $S^*(T - t)\, \omega(t)$ for their limits, and those vectors have finite norm. The norm of the first term remains finite as $\Delta \to 0$, by the reasoning of (2.30) and (2.31). Thus

$$\dot{q}(t) = \frac{-A^*}{\lambda^2} \int_t^T S^*(\tau - t)\, \omega(\tau)\, d\tau - \frac{1}{\lambda^2}\, \omega(t)$$

or

$$\dot{q}(t) = -A^* q(t) - \frac{1}{\lambda^2} (u_d(t) - u(t)) \qquad \| q(t) \| \to 0, \quad t \to T^-, \tag{2.34}$$

and the theorem is proved.

Combining the three theorems and the lemma of this section produces

THEOREM 2.4: A Necessary and Sufficient Condition for the Optimal Control and an Existence Theorem for the Solution of the Two-Point Boundary-Value Problem. *If*

I. *A is self-adjoint, $u_0 \in D(A)$, $u_d \in H(T)$, $u_d(t)$ continuous and $\in D(A)$, and $Au_d(t)$ Bochner-integrable, or*

II. *$u_0 \in D(A)$, $u_d(t) \in H(T)$, $u_d(t)$ is absolutely continuous and $\dot{u}_d(t)$ is continuous,*

then

(i) *A unique solution to the optimal control problem exists.*

(ii) *A sufficient condition for $f = B^*q$ to be the optimal control is that q satisfies the two-point boundary-value problem* (2.7).

(iii) *A unique solution to the two-point boundary-value problem exists.*

(iv) *Therefore, it is necessary that if $f = B^*q$ is the optimal control, q must satisfy the two-point boundary-value problem.*

(v) *The solution of the optimal control problem is equivalent to solving the two-point boundary-value problem.*

Before leaving this section, we note $D(A)$ is dense in H and that the set of absolutely continuous functions with continuous derivative is dense in $H(T)$ as is the set of continuous functions $\{v(t)\}$ such that $v(t) \in D(A)$ and $Av(t)$ Bochner-integrable. Therefore, the restrictions of Theorem 5.2.4 are mild ones.

If u_d should fail to meet the conditions of Theorem 2.3, for example, one can find a u_d' as close to u_d as desired for which the optimization problem has a solution provided by the two-point boundary-value problem. The engineering distinction between solving the optimization problem for u_d' instead of u_d is not significant.

For the optimal regulator problem, i.e., the optimal pursuit problem wherein $u_d = 0$, the two-point boundary-problem (2.7) has been previously developed as a necessary condition for the optimal control in Reference [5] using the techniques of the calculus of variations.

3. THE ALGORITHMIC SOLUTION OF A TWO-POINT BOUNDARY-VALUE PROBLEM

The previous formulation of the control problem was subject to the requirement that the linear bounded operator B have its range in $D(A)$. This may be too severe a restriction. In this section the restriction is removed, at least for systems where the semigroup is self-adjoint and compact, and both the class of admissible controls and the desired state is suitably constrained. B is taken as the identity operator. It is shown that the optimal control can again be obtained as the solution of the associated two-point boundary-value problem. Algorithms are given for the solution of the latter and hence, also, the optimal control problem. The restriction to compact self adjoint semigroups, with a pure point spectrum, it may be noted, is still general enough to include the classical self adjoint "Sturm–Liouville" problems of partial differential equations.

THEOREM 3.1. *The two-point boundary-value problem* [Eq. (3.1) below] *has a unique solution* $(u(t), q(t))$ *in* $H \times H$ *for* $t \in [0, T]$. $q(t)$ *is the optimal control and* $u(t)$ *is the optimal system trajectory, for the system described by* (3.2) (below) *provided:*

(1) A *is self-adjoint with a pure point spectrum whose eigenvalues are of finite multiplicity*;

(2) $S(t)$ *is a compact semigroup*;

(3) $u_d \in C$, *the class of functions* $\{v(t)\}$ *in* $H(T)$ *such that*

$$v(t) \in D(A), \qquad \int_0^T \| Av(t)\|^2 \, dt < \infty;$$

(4) *The admissible control class* $\{q(t)\} = C'$ *is in* $H(T)$ *and satisfies* $q(t) \in D(A)$ *and* $\int_0^T \| Aq(t) \| \, dt < \infty$;

(5) $u_0 \in D(A^2)$;

$$\left.\begin{aligned} \dot{u}(t) &= Au(t) + q(t); \qquad \| u(t) - u_0 \| \to 0, \quad t \to 0^+, \\ \dot{q}(t) &= -Aq(t) - \frac{1}{\lambda^2}(u_d(t) - u(t)); \qquad \| q(t)\| \to 0, \quad t \to T^-, \end{aligned}\right\} \tag{3.1}$$

$$\dot{u}(t) = Au(t) + q(t); \qquad \| u(t) - u_0 \| \to 0, \quad t \to 0^+. \tag{3.2}$$

Proof. The scheme of this proof is to show that under the stated conditions a unique solution $(u(t), q(t))$ to (3.1) exists, where $q(t)$ and $u(t)$ are in $D(A)$ and

$$\int_0^T \| Aq(t)\| \, dt < \infty \quad \text{and} \quad \int_0^T \| Au(t)\| \, dt < \infty.$$

Then since $D(A^2) \subset D(A)$, Theorem 2.1 may be paraphrased for $B = I$ to show that

$$q = (\lambda^2 I + G^* G) \, G^*(u_d - G_3 u_0).$$

But then q is the optimal control.

Let $\{\lambda_n\}$ be the eigenvalues of A and ϕ_n, the corresponding eigenfunctions orthonormalized so that $[\phi_i, \phi_j]_H = \delta_{ij}$. The using the finite multiplicity of the eigenvalues produces the representations for I, $S(t)$ and A shown below

$$y = Iy = \sum_n [\phi_n, y] \phi_n,$$

$$S(t)y = \sum_n e^{t\lambda_n}[\phi_n, y] \phi_n,$$

$$Az = \sum_n \lambda_n [\phi_n, z] \phi_n,$$

$$y \in H, \qquad z \in D(A), \qquad \lambda_n + 1 < \lambda_n < W_0 < \infty, \qquad \lambda_n \to -\infty. \tag{3.3}$$

Since the semigroup is compact (a compact operator for $t > 0$), $\{\lambda_n\}$ has no point of accumulation, except possibly the point at $-\infty$. Expressing $u_d(t)$ and u_0 in an eigenfunction series, and assuming a series solution $u(t)$, $q(t)$ for (3.1), produces

$$u_d(t) = \sum_{n=1}^{\infty} \beta_{n_d}(t) \phi_n(x),$$

$$u(t) = \sum_{n=1}^{\infty} \beta_n(t) \phi_n(x),$$

$$q(t) = \sum_{n=1}^{\infty} \alpha_n(t) \phi_n(x),$$

$$u_0 = \sum_{n=1}^{\infty} \beta_{n_0} \phi_n(x). \tag{3.4}$$

The condition that $u_d(t) \in D(A)$ is equivalent to

$$\sum_1^{\infty} \lambda_n^2 \beta_{n_d}^2(t) < \infty$$

for each $t > 0$ and the condition that $\int_0^T \| Au_d(t) \|_H^2 \, dt < \infty$ is equivalent to

$$\int_0^T \left[\sum_1^{\infty} \lambda_n^2 \beta_{n_d}^2(t) \right] dt < \infty.$$

The coefficients of the series (3.4) must satisfy

$$\frac{d}{dt} \begin{Bmatrix} \beta_n(t) \\ \alpha_n(t) \end{Bmatrix} = \begin{bmatrix} \lambda_n & 1 \\ \lambda^{-2} & -\lambda_n \end{bmatrix} \begin{Bmatrix} \beta_n(t) \\ \alpha_n(t) \end{Bmatrix} - \frac{1}{\lambda^2} \begin{Bmatrix} 0 \\ \beta_{n_d}(t) \end{Bmatrix}; \qquad \begin{matrix} \beta_n(0) = \beta_{n_0} \\ \alpha_n(T) = 0 \end{matrix} \tag{3.5}$$

or

$$\begin{Bmatrix}\beta_n(t)\\ \alpha_n(t)\end{Bmatrix} = \begin{bmatrix} \left(\cosh\gamma_n t + \frac{\lambda_n}{\gamma_n}\sinh\gamma_n t\right) & \frac{\sinh\gamma_n t}{\gamma_n} \\ \frac{\sinh\gamma_n t}{\gamma_n\lambda^2} & \left(\cosh\gamma_n t - \frac{\lambda_n}{\gamma^n}\sinh\gamma_n t\right)\end{bmatrix}\begin{Bmatrix}\beta_{n_0}\\ \alpha_n(0)\end{Bmatrix}$$

$$-\frac{1}{\lambda^2}\begin{Bmatrix} \int_0^t \frac{\sinh\gamma_n(t-\tau)}{\gamma_n}\beta_{n_d}(\tau)\,d\tau \\ \int_0^t \left(\cosh\gamma_n(t-\tau) - \frac{\lambda_n}{\gamma_n}\sinh\gamma_n(t-\tau)\right)\beta_{n_d}(\tau)\,d\tau\end{Bmatrix}$$

$$\gamma_n = (\lambda_n^2 + \lambda^{-2})^{1/2}$$

where $\alpha_n(0)$ is chosen so that $\alpha_n(T) = 0$;

$$\alpha_n(0) = \frac{-\sinh\gamma_n T\beta_{n_0}}{\gamma_n\lambda^2\left(\cosh\gamma_n T - \frac{\lambda_n}{\gamma_n}\sinh\gamma_n T\right)} + \frac{\int_0^T\left[\cosh\gamma_n(T-\tau) - \frac{\lambda_n}{\gamma_n}\sinh\gamma_n(T-\tau)\right]\beta_{n_d}(\tau)\,d\tau}{\lambda^2\left(\cosh\gamma_n T - \frac{\lambda_n}{\gamma_n}\sinh\gamma_n T\right)}. \tag{3.6}$$

Then

$$\beta_n(t) = \left[\frac{\lambda_n\gamma_n\sinh\gamma_n(t-T) + \gamma_n\cosh\gamma_n(t-T)}{\gamma_n^2\left(\cosh\gamma_n T - \frac{\lambda_n}{\gamma_n}\sinh\gamma_n T\right)}\right]\beta_{n_0}$$
$$+\frac{\sinh\gamma_n T\int_0^T\left[\cosh\gamma_n(T-\tau) - \frac{\lambda_n}{\gamma_n}\sinh\gamma_n(T-\tau)\right]\beta_{n_d}(\tau)\,d\tau}{\gamma_n^2\left(\cosh\gamma_n T - \frac{\lambda_n}{\gamma_n}\sinh\gamma_n T\right)}$$
$$-\frac{1}{\lambda^2}\int_0^t\frac{\sinh\gamma_n(t-\tau)}{\gamma_n}\beta_{n_d}(\tau)\,d\tau$$

$$\alpha_n(t) = \left[\frac{\sinh\gamma_n(t-T)}{\gamma_n\lambda^2\left(\cosh\gamma_n T - \frac{\lambda_n}{\gamma_n}\sinh\gamma_n T\right)}\right]\beta_{n_0}$$
$$+\frac{\left(\cosh\gamma_n t - \frac{\lambda_n}{\gamma_n}\sinh\gamma_n t\right)\int_0^T\left(\cosh\gamma_n(T-\tau) - \frac{\lambda_n}{\gamma_n}\sinh\gamma_n(T-\tau)\right)\beta_{n_d}(\tau)\,d\tau}{\left(\lambda^2\left[\cosh\gamma_n T - \frac{\lambda_n}{\gamma_n}\sinh\gamma_n T\right]\right)}$$
$$-\frac{1}{\lambda^2}\int_0^t\left(\cosh\gamma_n(t-\tau) - \frac{\lambda_n}{\gamma_n}\sinh\gamma_n(t-\tau)\right)\beta_{n_d}(\tau)\,d\tau. \tag{3.7}$$

Now consider the function

$$u(t) = \sum_1^\infty \beta_n(t)\,\phi_n(x),$$
$$q(t) = \sum_1^\infty \alpha_n(t)\,\phi_n(x), \tag{3.8}$$

where $\beta_n(t)$ and $\alpha_n(t)$ are defined as in (3.7). We will establish that $(u(t), q(t))$ is the solution of (3.1), that $q(t) \in D(A)$ and $\int_0^T \| Aq(t) \|_H^2\, dt < \infty$, and that $u(t) \in D(A)$ and $\int_0^T \| Au(t) \|_H^2\, dt < \infty$. This will prove the theorem.

$$\text{as} \quad n \to \infty, \qquad \lambda_n \to -\infty, \qquad \gamma_n \to \infty, \quad \text{and}$$
$$\sinh \gamma_n t \to \frac{e^{\gamma_n t}}{2} \to \frac{e^{-\lambda_n t}}{2},$$
$$\cosh \gamma_n t \to \frac{e^{-\lambda_n t}}{2},$$
$$\frac{\lambda_n}{\gamma_n} \to 1 + \frac{1}{2\lambda_n^2\lambda^2}, \tag{3.9}$$
$$\cosh \gamma_n t - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n t \to -\frac{1}{4\lambda_n^2\lambda^2}\, e^{-\lambda_n t}$$

and

$$\beta_n(t) \to e^{-\gamma_n t}\beta_{n_0} + \frac{1}{\gamma_n\lambda^2}\int_t^T e^{\gamma_n(t-\tau)}\,\beta_{n_d}(\tau)\,d\tau,$$
$$\alpha_n(t) \to -\frac{\lambda_n}{4}\, e^{-\gamma_n t}\beta_{n_0} - \frac{1}{4\lambda^4\lambda_n^2}\int_t^T e^{\gamma_n(t-\tau)}\,\beta_{n_d}(\tau)\,d\tau. \tag{3.10}$$

Thus, for large N

$$\begin{aligned}
\left(\sum_N^\infty \lambda_n^2\beta_n^2(t)\right)^{1/2} &\leqslant 2\left(\sum_N^\infty \lambda_n^2 e^{-2\gamma_n t}\beta_{n_0}^2\right)^{1/2} \\
&\quad + \frac{2}{\lambda^2}\left(\sum_N^\infty \left[\int_t^T e^{\gamma_n(t-\tau)}\,\beta_{n_d}(\tau)\,d\tau\right]^2\right)^{1/2} \\
&\leqslant 2\left(\sum_N^\infty \lambda_n^2\beta_{n_0}^2\right)^{1/2} + \frac{\sqrt{2}}{\lambda^4}\left(\sum_N^\infty \frac{1}{\gamma_n}\int_0^T \beta_{n_d}^2(\tau)\,d\tau\right)^{1/2} \\
&\leqslant \epsilon_N
\end{aligned} \tag{3.11}$$

because $u_0 \in D(A^2)$ and $\int_0^T \| Au_d(t) \|^2 \, dt < \infty$. Thus, the formal series solution for $u(t)$ is in $D(A)$. The bound ϵ_N in (3.11) is independent of t, and therefore

$$\int_0^T \| Au(t) \|^2 \, dt < \infty, \tag{3.12}$$

which implies that

$$\int_0^T \| Au(t) \| \, dt < \infty. \tag{3.13}$$

Similarly, we may show that $q(t)$ is in $D(A)$ and that

$$\int_0^T \| Aq(t) \| \, dt < \infty.$$

Finally, it remains to be shown that the formal solutions (3.4) may be termwise differentiated. From (3.11) we have that the series for $u(t)$ and $Au(t)$ are uniformly convergent in H. The series for $q(t)$ can be shown to also have these properties. Upon term-by-term differentiation of the series for $u(t)$, we have

$$\sum_1^\infty \dot{\beta}_n(t) \phi_n(x) = \sum_1^\infty \lambda_n \beta_n(t) \phi_n(x) + \sum_1^\infty \alpha_n(t) \phi_n(x). \tag{3.14}$$

The first term is the series for $Au(t)$ and the second the series for $h(t)$, and both of these are uniformly convergent with continuous coefficients. Therefore $u(t)$ is termwise differentiable. Similarly, $q(t)$ is termwise differentiable.

Thus, the solutions proposed (3.8) are valid solutions to the boundary-value problem and $q(t)$ is the optimal control. These solutions are unique because of the uniqueness of the eigenfunction representation, and this completes the proof. As a prelude to constructing an algorithm for the solution of the boundary-value problem (3.1), we consider some properties of the corresponding initial-value problem (3.15).

$$\begin{cases} \dot{u}(t) = Au(t) + q(t), & \| u(t) - u_0 \| \to 0, \quad t \to 0^+, \\ \dot{q}(t) = -Aq(t) - \dfrac{1}{\lambda^2}(u_d(t) - u(t)); & \| q(t) - q_0 \| \to 0, \quad t \to 0^+. \end{cases} \tag{3.15}$$

THEOREM 3.2. *If*

(1) *A is self-adjoint with a pure point spectrum whose eigenvalues are of finite multiplicity and $S(t)$ is compact*;

(2) $u_d(t), q_0 \in$ *Range* $S(2T)$, *for each* t, $u_0 \in$ *Range* $S(2T + \Delta)$, *for some* $\Delta > 0$;

then

(1) *the initial-value problem* (3.15) *has a solution* $(u(t), q(t))$ *such that*

$$u(t) \in \text{Range } S(T), \qquad q(t) \in \text{Range } S(T),$$

$$\int_0^T \| Au(t) \| \, dt < \infty, \qquad \int_0^T \| Aq(t) \| \, dt < \infty;$$

(2) $q(T) = Qq_0 - K$ *where* $K \in$ *Range* $S(T)$, *the linear mapping* Q *is self-adjoint, closed, and has a continuous inverse.*

$$D(Q) \text{ is dense in } H,$$
$$D(Q) \supset \text{Range } S(T),$$
$$D(Q^2) \supset \text{Range } S(2T);$$

(3) *there exists a unique* $q_0 = q_0' \in$ *Range* $S(2T)$ *for which the solution to the initial-value problem satisfies the boundary-value problem*;

(4) $Q\, S(2T)$ *is a compact operator.*

Proof. As in the preceding theorem, an eigenfunction series solution (3.4) is assumed and can be shown to be valid and have the required properties. The coefficients of the series for $u(t)$ and $q(t)$ are given below in (3.16) and (3.17), respectively.

$$\beta_n(t) = \left(\cosh \gamma_n t + \frac{\lambda_n}{\gamma_n} \sinh \gamma_n t\right) \beta_n(0) + \frac{\sinh \gamma_n t}{\gamma_n} \alpha_n(0) - \frac{1}{\lambda^2} \int_0^t \frac{\sinh \gamma_n (t-\tau)}{\gamma_n} b_{n_d}(\tau)\, d\tau, \tag{3.16}$$

$$\alpha_n(t) = \frac{\sinh \gamma_n t}{\gamma_n \lambda^2} \beta_n(0) + \left(\cosh \gamma_n t - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n t\right) \alpha_n(0) - \frac{1}{\lambda^2} \int_0^t \left(\cosh \gamma_n (t-\tau) - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n (t-\tau)\right) b_{n_d}(\tau)\, d\tau. \tag{3.17}$$

Equation (3.17) provides a representation for Q and K,

$$\begin{aligned} Qq(0) &= Q\left(\sum_1^\infty \alpha_n(0)\, \phi_n(x)\right) \\ &= \sum_1^\infty \left(\cosh \gamma_n T - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n T\right) \alpha_n(0)\, \phi_n(x) \\ &= \sum_1^\infty \left(\cosh \gamma_n T - \left(1 - \frac{1}{\gamma_n{}^2 \lambda^2}\right)^{1/2} \sinh \gamma_n T\right) \alpha_n(0)\, \phi_n(x), \end{aligned} \tag{3.18}$$

$$K = \sum - \frac{\sinh \gamma_n T}{\gamma_n \lambda^2} \beta_n(0) + \frac{1}{\lambda^2} \int_0^T \left(\cosh \gamma_n (t-\tau) - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n (t-\tau)\right) b_{n_d}(\tau)\, d\tau. \tag{3.19}$$

That Range $S(T) \subset D(Q)$ and Range $S(2T) \subset D(Q^2)$ and $K \in$ Range $S(T)$ can be directly shown using the asymptotic approximations (3.9). Also,

$$QS(2T)\left(\sum \alpha_n \phi_n(x)\right) = \sum_{n=1}^{\infty} \left(\cosh \gamma_n T - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n T\right) e^{2\lambda_n T} \alpha_n \phi_n(x)$$

$$= \sum_{n=1}^{\infty} \beta_n \phi_n(x), \tag{3.20}$$

$$\beta_n \to \frac{e^{\lambda_n T}}{2\lambda_n^2 \lambda^2} \alpha_n, \qquad \lambda_n \to -\infty. \tag{3.21}$$

Thus, $QS(2T)$ is compact, for it may be defined as a limit of a sequence of compact operators defined by an N-term truncation of the series above. The Range of $S(T)$ includes the set of all vectors in H represented by a finite eigenfunction series and this latter set is dense in H. Therefore, $D(Q)$ is dense in H. Q^{-1} is given by

$$Q^{-1}y = Q^{-1}\left(\sum_1^{\infty} a_n \phi_n(x)\right)$$

$$= \sum_1^{\infty} \left(\cosh \gamma_n T - \left(1 - \frac{1}{\gamma_n^2 \lambda^2}\right)^{1/2} \sinh \gamma_n T\right)^{-1} a_n \phi_n(x) \tag{3.22}$$

$\| Q^{-1} \|$ is bounded by

$$\sup_{\gamma_n \in [\lambda^{-1}, \infty]} \left[\cosh \gamma_n T - \left(1 - \frac{1}{\gamma_n^2 \lambda^2}\right)^{1/2} \sinh \gamma_n T\right]^{-1} < \infty$$

and Q, being the inverse of a bounded and hence closed operator, is closed.

In the course of the proof of Theorem 3.1 it was found that the optimal control problem was generated by the solution of (3.15) for

$$q_0 = q' = \sum_1^{\infty} \alpha_n'(0) \phi_n(x), \tag{3.23}$$

$$\alpha_n'(0) = \frac{-\sinh \gamma_n T \beta_{n_0}}{\gamma_n \lambda^2 \left(\cosh \gamma_n T - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n T\right)}$$
$$+ \frac{\int_0^T \left(\cosh \gamma_n (T - \tau) - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n (T - \tau)\right) \beta_{n_d}(\tau)\, d\tau}{\lambda^2 \left(\cosh \gamma_n T - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n T\right)}. \tag{3.24}$$

Thus, the fact that $u_d(t) \in$ Range $S(2T)$ and $u_0 \in$ Range $S(2T + \Delta)$ guarantees that $q_0' \in$ Range $S(2T)$. Since Q has an inverse, q_0' is unique. Q.E.D.

The work of this paper applies to discrete as well as distributed parameter systems. In the discrete case, H is the finite-dimensional Euclidean space E_N, A is a matrix operator, and $S(t) = e^{At}$ the matrix exponential operator. $S(t)$ has group properties and the solution to the initial condition problem (3.15) exists for any (u_0, q_0) in $H \times H$. But in the distributed case, as may be shown using the approach of Theorem 3.2, a solution to the initial-value problem need exist only for (u_0, q_0) restricted to certain subsets of $H \times H$. This is one sharp distinction between discrete and distributed parameter systems which plays a prominent role in the next theorem.

THEOREM 3.3: An algorithm which generates the optimal control. *If*

(1) *A is self-adjoint with a pure point spectrum whose eigenvalues are of finite multiplicity and the semigroup $S(t)$ is compact with empty null space,*

(2) *$u_d(t) \in$ Range $S(2T)$ for each t, $u_0 \in$ Range $S(2T + \Delta)$ for some $\Delta > 0$,*

(3) *the algorithm* (3.25) *is employed to calculate q_{0_N}, then the sequence of solutions q_n, u_n in* (3.26) (see below) *converge in $H(T)$ to the optimal control and optimal system trajectory, for the system and control constraint class C^1 defined by* (3.27) (below).

$$\begin{cases} V_{n+1} = V_n + \epsilon_n Z_n, \\ Z_n = S(2T) Q[QS(2T) V_n - K], \\ \epsilon_n = \dfrac{\| Z_n \|^2}{\| QS(2T) Z_n \|^2}, \\ q_{0_n} = S(2T) V_n, \\ V_1 \in H, \end{cases} \tag{3.25}$$

$$\begin{cases} \dot{u}_n(t) = Au_n(t) + q_n(t) & \| u_n(t) - u_0 \| \to 0, \quad t \to 0^+, \\ \dot{q}_n(t) = -Aq_n(t) - \dfrac{1}{\lambda^2}(u_d(t) - u_n(t)); & \| q_n(t) - q_{0_n} \| \to 0, \quad t \to 0^+, \end{cases} \tag{3.26}$$

$$\begin{gathered} \dot{u}(t) = Au(t) + q(t); \qquad \| u(t) - u_0 \| \to 0, \qquad t \to 0^+ \\ q(t) \in C', \text{ i.e., } q \in H(T), \qquad q(t) \in D(A) \text{ and } \int_0^T \| Aq(t) \| \, dt < \infty. \end{gathered} \tag{3.27}$$

Proof. From Theorem 3.2 it is known that for a unique $q_{0n} = q_0'$ in Range $S(2T)$, the solution to the initial-value problem (3.26) satisfies the corresponding boundary-value problem and therefore generates the optimal control. Equation (3.25) is a steepest-descent routine for minimizing

$$J(V) = \| Q\, S(2T)\, V - K \|^2, \tag{3.28}$$

the norm of $q(T)$, restricting our search to q_0 in the range of $S(2T)$. Since $K \in D(Q)$, Range $S(2T) \subset D(Q^2)$, it is equivalent to minimize

$$\begin{aligned} J'(V) &= [DV, V] - 2\,[V, K'], \\ D &= [Q\, S(2T)]^* \,[Q\, S(2T)] = S(2T) Q^2\, S(2T), \\ K' &= S(2T) QK. \end{aligned} \tag{3.29}$$

D is a compact self-adjoint operator, and a known solution V' exists to this problem of minimizing $J'(V)$,

$$S(2T)\, V' = q_0'. \tag{3.30}$$

With these conditions established, a proof that

$$\| Qq_{0_n} - K \| = \| q_n(T) \| \to 0 \tag{3.31}$$

may be found in Reference [*10*].

Since Q^{-1} is defined on all of H, the solution of the initial-value problem (3.26) is equivalent to the solution of the boundary-value problem (3.32),

$$\begin{cases} \dot{u}_n(t) = Au_n(t) + q_n(t), & \| u_n(t) - u_0 \| \to 0, \quad t \to 0^+, \\ \dot{q}_n(t) = -Aq_n(t) - \dfrac{1}{\lambda^2}\,(u_{\mathrm{d}}(t) - u_n(t)); & \| q_n(t) - q_n(T) \| \to 0, \quad t \to T^-. \end{cases} \tag{3.32}$$

Because $q_n(T)$ is in Range $S(T) \supset D(A^2)$ (Theorem 3.2), we can define

$$\hat{q}_n(t) = q_n(t) - q_n(T)$$

and replace (3.32) by

$$\begin{cases} \dot{u}_n(t) = Au_n(t) + q_n(t); & \| u_n(t) - u_0 \| \to 0, \quad t \to 0^+, \\ \dot{\hat{q}}_n(t) = -A\hat{q}_n(t) - \dfrac{1}{\lambda^2}\,[u_{\mathrm{d}}(t) - u_n(t)] - Aq_n(T); & \| \hat{q}_n(t) \| \to 0, \quad t \to T^-. \end{cases} \tag{3.33}$$

Now $q_n(t)$, $\hat{q}_n(t)$, $u_{\mathrm{d}}(t)$ and $Aq_n(T)$ are all in H and therefore, using results (2.11) and (2.12) of Theorem 2.1 for $B = I$,

$$\begin{cases} u_n(t) = S(t)\, u_0 + \displaystyle\int_0^t S(t - \tau)\, q_n(\tau)\, d\tau, \\ \hat{q}_n(t) = \displaystyle\int_t^T d\tau\, S(\tau - t) \left[\frac{1}{\lambda^2}\,(u_{\mathrm{d}}(\tau) - u_n(\tau)) + Aq_n(T) \right] \end{cases} \tag{3.34}$$

since

$$\int_t^T S(\tau - t)\, Aq_n(T)\, d\tau = \int_t^T \frac{d}{d\tau}\,[S(\tau - t)\, q_n(T)]\, d\tau$$
$$= S(T - t)\, q_n(T) - q_n(T),$$

$$\begin{cases} u_n(t) = S(t)\, u_0 + \int_0^t S(t - \tau)\, q_n(\tau)\, d\tau \\ q_n(t) = S(T - t)\, q_n(T) + \dfrac{1}{\lambda^2} \int_t^T d\tau\, S(\tau - t)[u_{\mathrm{d}}(\tau) - u_n(\tau)] \end{cases} \tag{3.35}$$

or

$$q_n(t) = S(T - t)\, q_n(T) + \frac{1}{\lambda^2} \int_t^T d\tau [S(\tau - t)\, u_{\mathrm{d}}(\tau) - S(\tau)\, u_0]$$
$$- \frac{1}{\lambda^2} \int_t^T d\tau\, S(\tau - t) \int_0^\tau d\sigma\, S(\tau - \sigma)\, q_n(\sigma) \tag{3.36}$$

In the notation of eqs. (2.3) and (2.4),

$$q_n = (G^*G + \lambda^2 I)^{-1}\, S(T - \cdot)\, q_n(T) + (G^*G + \lambda^2 I)^{-1}\, G^*(u_{\mathrm{d}} - G_1 u_0). \tag{3.37}$$

The second term above is recognized as the optimal control and the first term is seen to be a continuous mapping of $q_n(T)$ into $H(T)$. The fact that $q_n(T) \to 0$ establishes that the sequence of controls q_n which arise in the algorithm approach the optimal control in the $H(T)$ norm. The system trajectory depends continuously on q, and therefore u_n approaches the optimal system trajectory.

It is emphasized that the operations required in the preceding algorithm are all simply mechanized. $S(2T)z$ is simply the solution at time $2T$ of the unforced system equation with initial condition $z \in D(A)$, and Qy is the $q(T)$ solution of the initial-condition problem (3.26) for $q_0 = y \in S(T)$ and $u_{\mathrm{d}}(t) = u_0 = 0$. The restrictions placed on u_{d}, q_0 and u_0 are not severe in that these elements may be chosen from sets dense in their respective spaces.

4. An Example

As an example, the optimal control for a one dimensional diffusion equation is derived by solving the associated two-point boundary-value problem. The system equation is given by

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + q(x, t) \qquad \text{for} \quad 0 < t < \infty \quad \text{and} \quad 0 < x < 1, \tag{4.1}$$
$$u(x, 0) = u_0 = 0,$$
$$u(0, t) = u(1, t) = 0 \qquad \text{for} \quad t > 0,$$

The control problem is phrased using the notation of this paper as follows. Let $H = \mathscr{L}_2[0, 1]$,

$$\begin{aligned}
&\frac{\partial u(t)}{\partial t} = Au(t) + q(t) \qquad \text{for} \quad t > 0 \\
&D(A) = \{g(x) \in H\colon g(0) = g(2\pi) = 0; g(x), g'(x) \\
&\qquad\qquad \text{absolutely continuous and}, g''(x) \in H\}, \\
&Ag(x) = g''(x), \qquad g \in D(A), \\
&u_0 = 0 \in D(A), \\
&\| u(t) - u_0 \|_H \to 0 \quad \text{as} \quad t \to 0^+ \\
&q(t) \in D(A), \qquad \int_0^T dt \int_0^1 dx\, q^2(x, t) < \infty, \\
&\qquad \int_0^T dt \left[\int_0^1 dx \left[\frac{\partial^2}{\partial x^2} q(x, t)\right]^2\right]^{1/2} < \infty.
\end{aligned} \tag{4.2}$$

A is linear on $D(A)$ and, being a differential operator, is closed. The space of functions in $\mathscr{L}_2[0, 1]$ having an absolutely continuous first derivative, and having a second derivative in $\mathscr{L}_2[0, 1]$ is dense in $\mathscr{L}_2[0, 1]$. Therefore $D(A)$ is dense in H. Any f in H has the Fourier series representation

$$\begin{aligned}
f(x) &= 2 \sum_{n=1}^{\infty} [\sin n\pi x, f(x)]_H \sin n\pi x \\
&= \sum_{n=1}^{\infty} a_n \sin n\pi x.
\end{aligned} \tag{4.3}$$

Therefore, for $\alpha \neq -n^2\pi^2$,

$$(\alpha - A)^{-1} f = \sum_1^{\infty} a_n \frac{\sin \eta\pi x}{\alpha + n^2\pi^2}. \tag{4.4}$$

Hence

$$\|(\alpha - A)^{-1}\|_H^2 < \frac{1}{\alpha} \qquad \text{for} \quad \alpha > 0. \tag{4.5}$$

This condition, plus the fact that A is a closed linear operator with domain dense in H, is sufficient to establish that A is the infinitesimal generator of a strongly continuous semigroup. See, for example, the corollary to the Hille–Yosida–Phillips theorem in Reference 8. The theory of the preceding sections may now be employed. Prior to doing this observe that A has a pure point spectrum with eigenvalues $\lambda_n = -n^2\pi^2$ for $n > 0$, and corresponding orthonormal eigenvectors $\phi_n = \sqrt{2} \sin n\pi x$.

A is self-adjoint and generates the compact self-adjoint semigroup $S(t)$, where

$$\begin{aligned} S(t)f &= S(t)\Big(\sum_{n=1}^{\infty} a_n \sin n\pi x\Big) \\ &= \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \sin n\pi x. \end{aligned} \tag{4.6}$$

The optimization problem under consideration is that of choosing $q(x, t)$, in the class defined in (4.2), so as to minimize

$$J(q) = \int_0^T dt \int_0^1 dx\, [u_{\mathrm{d}}(x, t) - u(x, t)]^2 + \lambda^2 \int_0^T dt \int_0^1 dx\, [q(x. t)]^2, \tag{4.7}$$

where

$$\begin{aligned} &u_{\mathrm{d}}(t) \in D(A), \\ &\int_0^T dt \int_0^1 dx\, [u_{\mathrm{d}}(x, t)]^2 < \infty, \\ &\int_0^T dt \int_0^1 dx \left[\frac{\partial^2}{\partial x^2} u_{\mathrm{d}}(x, t)\right]^2 < \infty. \end{aligned} \tag{4.8}$$

Theorem 3.1 asserts that an optimal control exists uniquely and is given by the solution to (5.9)

$$\begin{cases} \dfrac{\partial u(x, t)}{\partial t} = \dfrac{\partial^2 u(x, t)}{\partial x^2} + q(x, t), \\ \dfrac{\partial q(x, t)}{\partial t} = -\dfrac{\partial^2 q(x, t)}{\partial x^2} - \dfrac{1}{\lambda^2}[u_{\mathrm{d}}(x, t) - u(x, t)], \\ u(0, t) = u(1, t) = q(0, t) = q(1, t) = u(x, 0) = q(x, T) = 0. \end{cases} \tag{4.9}$$

In fact, (4.9) has already been solved in more abstract form as part of the proof of Theorem 3.1. There it is shown that if $q(x, t)$ and $u_{\mathrm{d}}(x, t)$ are written in their eigenfunction representations

$$\begin{aligned} u_{\mathrm{d}}(x, t) &= 2 \sum_{n=1}^{\infty} \Big(\int_0^1 d\sigma \sin \eta\pi\sigma\, u_{\mathrm{d}}(\sigma, t)\Big) \sin n\pi x \\ &= \sqrt{2} \sum_{n=1}^{\infty} \beta_{n_{\mathrm{d}}}(t) \sin n\pi x, \end{aligned} \tag{4.10}$$

$$q(x, t) = \sqrt{2} \sum_{n=1}^{\infty} \alpha_n(t) \sin n\pi x, \tag{4.11}$$

then

$$\alpha_n(t) = \frac{\left(\cosh \gamma_n t - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n t\right) Z_n}{\lambda^2 \left(\cosh \gamma_n T - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n T\right)} - \frac{1}{\lambda^2} \int_0^t \left[\cosh \gamma_n(t-\tau) - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n(t-\tau)\right] \beta_{n_d}(\tau)\, d\tau \tag{4.12}$$

where

$$\begin{aligned} \lambda_n &= -n^2\pi^2 \\ \gamma_n &= (n^4\pi^4 + \lambda^{-2})^{1/2} \\ Z_n &= \int_0^T \left[\cosh \gamma_n(T-\tau) - \frac{\lambda_n}{\gamma_n} \sinh \gamma_n(T-\tau)\right] \beta_{n_d}(\tau)\, d\tau. \end{aligned} \tag{4.13}$$

The optimal control has therefore been determined.

From this discussion surrounding (4.6) it is clear that A is self-adjoint with a pure point spectrum whose eigenvalues are of finite multiplicity and that the semigroup $S(t)$ is compact with empty null space. Therefore, if we restrict $u_d(t)$ to be in the range of $S(2T)$, i.e., if we require that

$$U_d(t) = \sum_{n=1}^{\infty} \beta'_{n_d}(t)\, e^{-2n^2\pi^2 T} \sin \pi\chi \tag{4.14}$$

where

$$\sum_{n=1}^{\infty} \beta'^2_{n_d}(t) < \infty \qquad \text{for} \quad t \in [0, T],$$

then the conditions of Theorem 3.3 are satisfied and the algorithm of that theorem may be used to calculate the optimal control. For this example, the algorithm takes the form of the following sequence of initial-condition problems.

1. Choose $V_1 \in S_2[0, 1]$, say $V_1(\chi) = 0$.
2. Set $n = 1$.
3. Solve

$$\frac{\partial W(\chi, t)}{\partial t} = \frac{\partial^2 W(\chi, t)}{\partial \chi^2}, \qquad 0 < t \leqslant 2T;\ W(\chi, t) \to 0 \text{ as } \chi \to 0^+ \text{ and } \chi \to 1^-$$
$$W(\chi, t) \to V_n(\chi) \text{ as } t \to 0^+.$$

4. Solve

$$\begin{cases} \dfrac{\partial V(\chi,t)}{\partial t} = \dfrac{\partial^2 V(\chi,t)}{\partial \chi^2} + y(\chi,t), & 0 < t \leqslant T;\ V(\chi,t) \to 0 \text{ as } \chi \to 0^+ \text{ and } \chi \to 1^- \\ & V(\chi,t) \to 0 \text{ as } t \to 0^+, \\ \dfrac{\partial y(\chi,t)}{\partial t} = \dfrac{-\partial^2 y(\chi,t)}{\partial \chi^2} - \dfrac{1}{\lambda^2}(U_{\mathrm{d}}(\chi,t) - V(\chi,t)); & y(\chi,t) \to 0 \text{ as } \chi \to 0^+ \text{ and } \chi \to {}^-1 \\ & y(\chi,t) \to W(\chi,2T) \text{ as } t \to 0^+. \end{cases}$$

5. Solve

$$\begin{cases} \dfrac{\partial V'(\chi,t)}{\partial t} = \dfrac{\partial V'^2(\chi,t)}{\partial \chi^2} + y'(\chi,t), & 0 < t \leqslant T;\ V'(\chi,t) \to 0 \text{ as } \chi \to 0^+ \text{ and } \chi \to 1^- \\ & V'(\chi,t) \to 0 \text{ as } t \to 0^+, \\ \dfrac{\partial y'(\chi,t)}{\partial t} = \dfrac{-\partial^2 y'(\chi,t)}{\partial \chi^2} + \dfrac{1}{\lambda^2} V'(\chi,t); & y'(\chi,t) \to 0 \text{ as } \chi \to 0^+ \text{ and } \chi \to 1^- \\ & y'(\chi,t) \to y(\chi,T) \text{ as } t \to 0^+. \end{cases}$$

6. Solve the equation in 3 for initial condition $W(\chi, t) \to V'(\chi, t)$ as $t \to 0^+$. Call the new solution $Z_n(\chi, t)$. Define $Z_n = Z_n(\chi, 2T)$.

7. Solve Eq. 3 for initial condition $W(\chi, t) \to Z_n$ as $t \to 0^+$. Call the new solution $W'(\chi, t)$.

8. Solve equations in 5 for initial condition $y'(\chi, t) \to W'(\chi, 2T)$ for $t \to 0^+$ Call the new solution $y_n'(\chi, t)$.

9. Evaluate

$$\| Z_n \|^2 = \int_0^1 Z_n^2(\chi, 2T)\, d\chi.$$

and

$$\| y_n' \|^2 = \int_0^1 (y_n'(\chi, T))^2\, d\chi.$$

10. Define

$$V_{n+1} = V_n + \frac{\| Z_n \|^2}{\| y_n' \|^2} \quad Z_n .$$

11. Set $n = n + 1$ and iterate by returning to step 3 of the sequence. In the nth cycle of this sequence, the estimate $q_n(\chi, t)$ of the optimal control is given by the solution $y(\chi, t)$ of the fourth step. As we have shown in Theorem 3.3, $q_n(\chi, t)$ converges to the optimal control.

5. Conclusions

This paper has presented sufficiency conditions for the solution of an optional control problem for a class of distributed parameter systems as well as existence conditions for the solution of the associated two-point boundary-value problem. By specializing the class of problems, an algorithm was developed in which a sequence of initial-value problems was solved in each cycle. It was shown that a pair of solutions produced in the cycle converge, respectively, to the optimal control and optimal system trajectory. The performance of this algorithm in the presence of numerical computational errors is unknown and is one direction future study of this subject could take.

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